Application Of Integrals

Unit 8

SECTION – A

Questions 1 to 10 carry 1 mark each.

1. Area of the region bounded by the curve $y = \sqrt{49 - x^2}$ and the x-axis is

(*a*) $\frac{49}{2}\pi$ sq units (*b*) 98π sq units (*c*) 49π sq units (*d*) 240π sq units

Ans. (*a*) $\frac{49}{2}\pi$ sq units

as area is above the x-axis

$$\therefore \quad \text{area} = 2\int_0^7 \sqrt{49 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}\frac{x}{7}\right]_0^7$$

$$= 2\left[\left(\frac{7}{2}\times 0 + \frac{49}{2}\sin^{-1}1\right) - (0)\right]$$

$$= \frac{49}{2}\pi \text{ sq units}$$

2. Area of the region bounded by the curve $x = 2y + 3$, the y-axis and between $y = -1$ and $y = 1$ is

(*a*) 4 sq units (*b*) $\frac{3}{2}$ sq units (*c*) 6 sq units (*d*) 8 sq units

Ans. (*c*) 6 sq units

$$\text{area} = \int_{-1}^{1} (2y + 3)\,dy = 6 \text{ sq units}$$

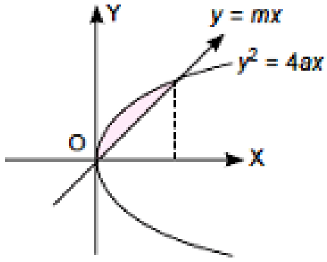
3. If the area bounded by the curves $y^2 = 4ax$ and $y = mx$ is $\frac{a^2}{3}$, then the value of m is

(*a*) 2 (*b*) –2 (*c*) $\frac{1}{2}$ (*d*) none of these

Ans. (*a*) 2

$$(mx)^2 = 4ax \Rightarrow m^2x^2 = 4ax \Rightarrow x = 0,\ x = \frac{4a}{m^2}$$

as the two curves intersect at 0, $\frac{4a}{m^2}$



$\therefore$ Area $= \int_0^{\frac{4a}{m^2}} (\sqrt{4ax} - mx)dx$

$= \left[2\sqrt{a} \cdot \frac{2}{3}x^{3/2} - \frac{mx^2}{2}\right]_0^{\frac{4a}{m^2}}$

$= \frac{4\sqrt{a}}{3} \cdot \frac{4a}{m^2} \cdot \frac{2\sqrt{a}}{m} - \frac{m}{2} \cdot \frac{16a^2}{m^4}$

$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}$

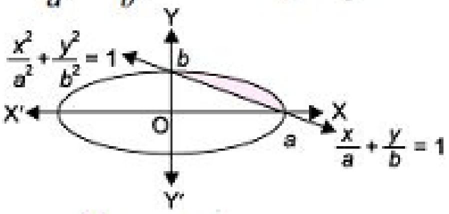
Given $\frac{8a^2}{3m^3} = \frac{a^2}{3} \Rightarrow m^3 = 8 \Rightarrow m = 2.$

4. The area of the smaller region between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$ in first quadrant is

(*a*) $\frac{1}{2}\ ab$ (*b*) $\frac{1}{2}\ \pi\ ab$ (*c*) $\pi\ ab$ (*d*) $\frac{ab}{4}\ (\pi - 2)$

Ans. (*d*) $\frac{ab}{4}\ (\pi - 2)$

y_1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $y_2 = \frac{x}{a} + \frac{y}{b} = 1$



Area $= \int_0^a (y_1 - y_2)\, dx$

$= \int_0^a \left\{\frac{b}{a}\sqrt{a^2 - x^2} - \frac{b}{a}(a - x)\right\}dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - ax + \frac{x^2}{2}\right]_0^a$

$= \frac{b}{a}\left[\left(0 + \frac{a^2}{2} \cdot \sin^{-1}1 - a^2 + \frac{a^2}{2}\right) - 0\right]$

$= \frac{b}{a}\left[\frac{a^2}{2} \cdot \frac{\pi}{2} - \frac{a^2}{2}\right] = \frac{b}{a} \cdot \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$

$= \frac{ab}{4}(\pi - 2)$ sq units

5. Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

(*a*) $16\ \pi$ (*b*) $4\ \pi$ (*c*) $32\ \pi$ (*d*) none of these

Ans. (*b*) $4\ \pi$

6. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(*a*) 2 (*b*) $\frac{9}{4}$ (*c*) $\frac{9}{3}$ (*d*) $\frac{9}{2}$

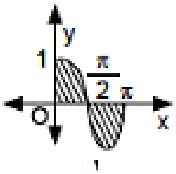
Ans. (*b*) $\frac{9}{4}$

7. Area bounded by the curve y = cos x, the x-axis and between x = 0, x = π is

(a) 4 sq units (b) 0 sq units (c) 1 sq unit (d) 2 sq units

Ans. (d) 2 sq units

Area $= \int_0^{\pi} |\cos x| dx$



$= \int_0^{\frac{\pi}{2}} \cos x\, dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\, dx$

$= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}$

$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\pi - \sin\frac{\pi}{2}\right]$

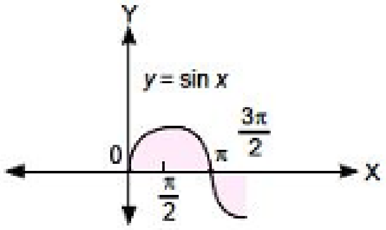
$= 1 - 0 - 0 + 1 = 2$ sq units

8. Area of the region bounded by the curve y = sin x between x = 0 and x = $\frac{3\pi}{2}$ is

(a) 3 sq units (b) 4 sq units (c) 5 sq units (d) π sq units

Ans. (a) 3 sq units

As y = sin x



Required area $= \int_0^{\pi} \sin x\, dx + \left|\int_{\pi}^{\frac{3\pi}{2}} \sin x\, dx\right|$

$= \left[-\cos x\right]_0^{\pi} + \left|\left[-\cos x\right]_{\pi}^{\frac{3\pi}{2}}\right|$

$= 2 + 1 = 3$ sq units

In the following questions 9 and 10, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:

(a) Both Assertion (A) and Reason (R) are true and Reason(R) is the correct explanation of assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason(R) is not the correct explanation of assertion (A).

(c) Assertion (A) is true but reason (R) is false.

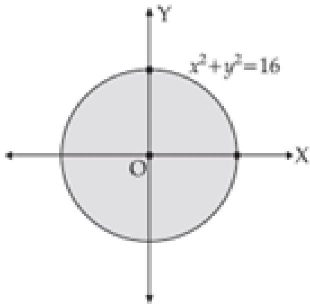
(d) Assertion (A) is false but reason (R) is true.

9. **Assertion (A):** The region bounded by the curve $y^2 = 16x$, Y-axis and the line y = 2 is 8/3.

Reason (R): Required area = $\int_0^2 x\,dy$

Ans. (d) Assertion (A) is false but reason (R) is true.

10. Assertion (A): The area bounded by the circle $x^2 + y^2 = 16$ is 16π sq. units.
Reason (R): We have $x^2 + y^2 = 16$, which is a circle having centre at (0, 0) and radius 4 units.
$\therefore y^2 = 16 - x^2 \Rightarrow y = \sqrt{16 - x^2}$



From figure, area of shaded region, $A = 4\int_0^4 \sqrt{16 - x^2}\,dx$

Ans. (b) Both Assertion (A) and Reason (R) are true but Reason(R) is not the correct explanation of assertion (A).

SECTION – B

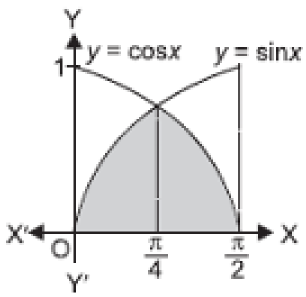
Questions 11 to 14 carry 2 marks each.

11. Find the area of the region bounded by the curve $y = \frac{1}{x}$, x-axis and between $x = 1$, $x = 4$.

Ans. Curve is $y = \frac{1}{x}$, x-axis and between $x = 1$, $x = 4$

Area = $\int_1^4 \frac{1}{x} dx$ = $[\log |x|]_1^4$ = log 4 – log 1 = log 4 sq units.

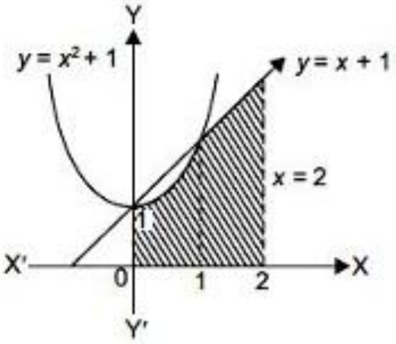
12. Write an expression for finding the area bounded by the curves $y = \sin x$ and $y = \cos x$, between $x = 0$, $x = \frac{\pi}{2}$ and the x -axis.



Ans. For the shaded area, the two curves intersect at $x = \frac{\pi}{4}$ (as $\sin x = \cos x$)

$\therefore$ area = $\int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$

13. Find the area of the region $\{(x, y): 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.
Ans.



$$\text{Area} = \int_0^1 (x^2+1)dx + \int_1^2 (x+1)\,dx$$

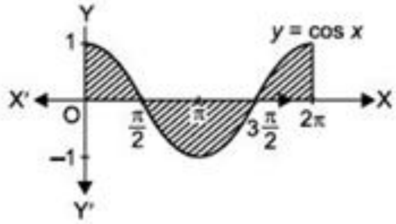
$$= \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2$$

$$= \left[\left(\frac{1}{3}+1\right)-0\right]+\left[\left(\frac{4}{2}+2\right)-\left(\frac{1}{2}+1\right)\right]$$

$$= \frac{23}{6} \text{ sq units}$$

14. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.

Ans.



$$\text{Area} = \int_0^{\frac{\pi}{2}} \cos x\, dx + \left| \int_{\frac{\pi}{2}}^{3\frac{\pi}{2}} \cos x\, dx \right| + \int_{3\frac{\pi}{2}}^{2\pi} \cos x\, dx$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} + \left|\sin x\right|_{\frac{\pi}{2}}^{3\frac{\pi}{2}} + \left[\sin x\right]_{3\frac{\pi}{2}}^{2\pi}$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) + \left|\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right)\right| + \left(\sin 2\pi - \sin\frac{3\pi}{2}\right)$$

$$= (1-0) + |(-1-1)| + \{(0-(-1)\}$$

$$= 4 \text{ sq units}$$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Draw a sketch of the following region and find its area:

$\{(x, y) : x2 + y2 \le 1 \le x + y\}$.

Ans.

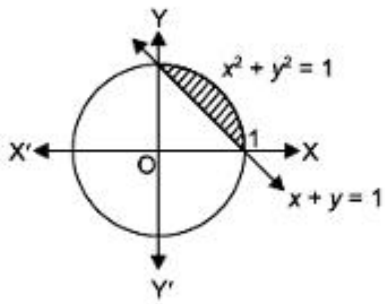
Curves are $x^2 + y^2 = 1$ and $x + y = 1$

$$\text{Area bounded} = \int_0^1 \{\sqrt{1-x^2} - (1-x)\}dx$$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - x + \frac{x^2}{2}\right]_0^1$$

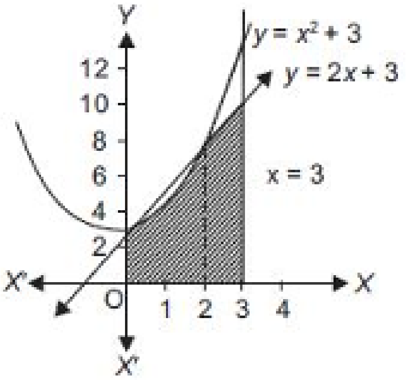
$$= \left(0 + \frac{1}{2}\sin^{-1}1 - 1 + \frac{1}{2}\right) - 0$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2} = \frac{1}{4}(\pi - 2) \text{ sq units}$$



16. Make a rough sketch of the region given below and find its area using integration $\{(x, y) : 0 \le y \le x^2 + 3; 0 \le y \le 2x + 3, 0 \le x \le 3\}$.

Ans.



Region = $\{(x, y): 0 \le y \le x^2 + 3, 0 \le y \le 2x + 3, 0 \le x \le 3\}$

On plotting the inequations we have to find the area of the shaded portion

Eliminating y from corresponding equations, we get

$x^2 + 3 = 2x + 3 \Rightarrow x = 0, 2$

$$\therefore \text{ area } = \int_0^2 (x^2 + 3)\, dx + \int_2^3 (2x + 3)\, dx.$$

$$= \left[\frac{x^3}{3} + 3x\right]_0^2 + [x^2 + 3x]_2^3$$

$$= \left(\frac{8}{3} + 6\right) - (0) + (9 + 9) - (4 + 6)$$

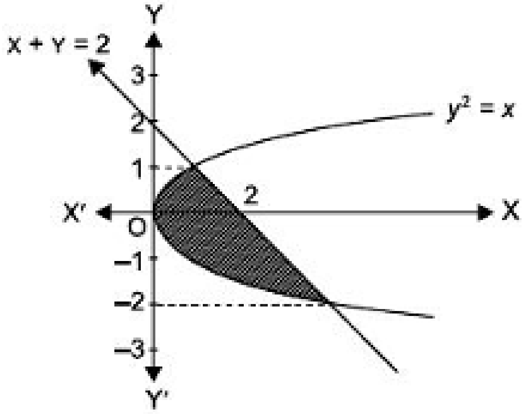
$$= \left(\frac{8}{3} + 6 + 18 - 10\right) \text{ sq units} = \frac{50}{3} \text{ sq units}$$

17. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

Ans.

Parabola is $y^2 = x$ and the line is $x + y = 2$

We have to find the shaded area.



Eliminating x, we get $y^2 + y - 2 = 0$

$\Rightarrow (y + 2)(y - 1) = 0 \quad \Rightarrow \quad y = -2, 1$

$$\therefore \text{ area } = \int_{-2}^{1} \{(2 - y) - y^2\}\, dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= \left(2 - \frac{5}{6} + 6 - \frac{8}{3}\right) = \left(8 - \frac{5}{6} - \frac{16}{6}\right)$$

$$= \frac{27}{6} = \frac{9}{2} \text{ sq units}$$

SECTION – D

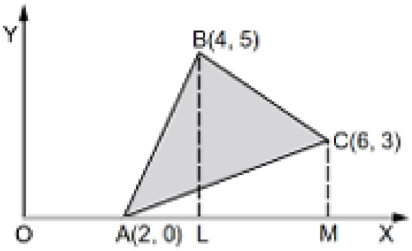
Questions 18 carry 5 marks.

18. Using integration, find the area of ΔABC, whose vertices are A(2, 0), B(4, 5) and C(6, 3).

Ans:

The equation of side AB is

$$\frac{y-0}{x-2}=\frac{(5-0)}{(4-2)} \Rightarrow y=\frac{5}{2}(x-2) \quad \dots \text{(i)}$$



The equation of side BC is

$$\frac{y-5}{x-4}=\frac{(3-5)}{(6-4)} \Rightarrow y=-x+9 \quad \dots \text{(ii)}$$

The equation of side AC is

$$\frac{y-0}{x-2}=\frac{(3-0)}{(6-2)} \Rightarrow y=\frac{3}{4}(x-2) \quad \dots \text{(iii)}$$

Draw perpendiculars BL and CM on the x-axis.

$\therefore$ area of $\triangle ABC$

$$= \text{ar}(\triangle ALB) + \text{ar}(\text{trap. } BLMC) - \text{ar}(\triangle AMC)$$

$$=\int_2^4 y_{AB}\,dx+\int_4^6 y_{BC}\,dx-\int_2^6 y_{AC}\,dx=\frac{5}{2}\int_2^4 (x-2)\,dx+\int_4^6 (9-x)\,dx-\frac{3}{4}\int_2^6 (x-2)\,dx$$

$$=\frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4+\left[9x-\frac{x^2}{2}\right]_4^6-\frac{3}{4}\cdot\left[\frac{x^2}{2}-2x\right]_2^6$$

$$=\frac{5}{2}[0-(-2)]+(36-28)-\frac{3}{4}[6-(-2)]=(5+8-6)\text{ sq units}=7\text{ sq units.}$$

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. Case-Study 1: Read the following passage and answer the questions given below.

A mirror in the shape of an ellipse represented by $\frac{x^2}{9}+\frac{y^2}{4}=1$ was hanging on the wall. Arun and his sister were playing with ball inside the house, even their mother refused to do so. All of sudden, ball hit the mirror and got a scratch in the shape of line represented by $\frac{x}{3}+\frac{y}{2}=1$

Based on the above information, answer the following questions.

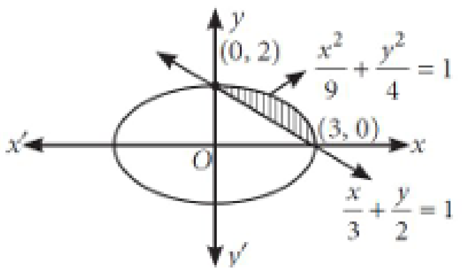
(a) Find the point(s) of intersection of ellipse and scratch (straight line). [1]

(b) Draw the figure which represents the Area of smaller region bounded by the ellipse and line. [1]

(c) Find the value of $\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx$ [2]

Ans. (a) Points (0, 2) and (3, 0) pass through both line and ellipse.

(b)



(c)

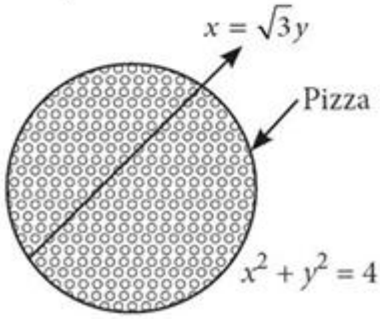
$$\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx = \frac{2}{3}\int_0^3 \sqrt{(3)^2-x^2}\,dx$$

$$= \frac{2}{3}\left[\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{2}\sqrt{0} + \frac{9}{2}\sin^{-1}(1) - \frac{1}{2}(0) - \frac{9}{2}\sin^{-1}(0)\right] = \frac{2}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi}{2}$$

20. Case-Study 2: Read the following passage and answer the questions given below.

A child cut a pizza with a knife. Pizza is circular in shape which is represented by $x^2 + y^2 = 4$ and sharp edge of knife represents a straight line given by $x = \sqrt{3}y$.



Based on the above information, answer the following questions.

(a) Find the point(s) of intersection of the edge of knife (line) and pizza shown in the figure [2]

(b) Find the value of area of the region bounded by circular pizza and edge of knife in first quadrant [2]

Ans. (a) We have, $x^2 + y^2 = 4$ ...(i)

and $x = \sqrt{3}y$ ...(ii)

From (i) and (ii), we get $3y^2 + y^2 = 4 \Rightarrow 4y^2 = 4 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

From (ii), $x = \sqrt{3}, -\sqrt{3}$

$\therefore$ Points of intersection of pizza and edge of knife are $(\sqrt{3}, 1), (-\sqrt{3}, -1)$.

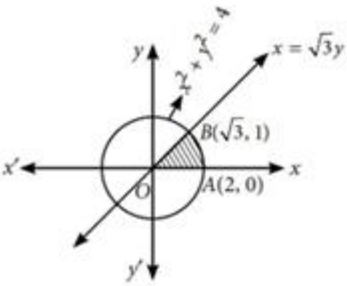
(b)

$$\text{Required area} = \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^2$$

$$= \frac{1}{\sqrt{3}}\left[\frac{3}{2} - 0\right] + \left[2\sin^{-1}(1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{2\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq. units}$$



....